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Eddy Viscosity Model for Turbulent Pipe Flow

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I. Introduction

A UNIFIED viscosity model will be sought here for the entire pipe with the requirement that in order to qualify as an acceptable model it must yield not only a mean velocity distribution, but also a turbulent shear, turbulent energy production rate, and direct viscous dissipation rate, in close agreement with experimental data. This aim, which certainly falls short of the task of explaining the complete structure of turbulence in a pipe, can provide a stronghold on some of the more significant and practical aspects of the mean flow properties.

II. Analysis

1) Overlap region

In his paper, Millikan[1] proposed a derivation of the celebrated log law which was based on a minimum set of a priori assumptions. His derivation proceeded as follows: If in the wall region we accept the law of the wall, $u^+ = f(y^+)$ and in the core region the velocity defect law $u_{\max}{}^+ - u^+ = g(\eta)$, by assuming the existence of an overlap region Σ the only functions f and g which are simultaneously satisfied in Σ are

$$f = u^+ = (1/k_1)\ \ln y^+ + k_2' \tag{1}$$

$$g = u_{\max}{}^+ - u^+ = -(1/k_1)\ \ln\eta + k_2'' \tag{1a}$$

and consistent with these equations the maximum velocity is given by

$$u_{\max}{}^+ = -(1/k_1)\ \ln\delta_\tau + (k_2' + k_2'') \tag{1b}$$

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In Eq. (1), k_1, k_2', k_2'' are arbitrary constants, $u^+ = u/u_\tau$, $u_{\max}{}^+ = u_{\max}/u_\tau$, $y^+ = yu_\tau/\nu$, $\eta = y/r$, where u_τ is the friction velocity $(\tau_w/\rho)^{1/2}$, r the radius of the pipe and $\delta_\tau = (u_\tau r/\nu)^{-1}$ is the inverse of the Reynolds number based on the friction velocity. Equations (1), (1a) and (1b) may be considered as the limit of the law of the wall, the defect law and the friction law for large Reynolds numbers, or equivalently for small δ_τ.

Introducing the definition of the eddy viscosity in the form $\tau_t = -\langle\rho u'v'\rangle = \rho\epsilon du/dy$ where τ_t is the turbulent shear, into the equation of motion $\tau = \mu du/dy - \langle\rho u'v'\rangle$ yields after nondimensionalization,

$$\tau^+ = \tau/\tau_w = (\tilde\epsilon + \delta_\tau)du^+/d\eta \tag{2}$$

where $\tilde\epsilon = \epsilon/u_\tau r$.

Taking the limit of Eq. (2) as $\delta_\tau \to 0$, granting that ϵ^+ possesses a regular limit, and utilizing the result obtained for $du^+/d\eta$ in Eq. (1a) gives the eddy viscosity at the limit of infinite Reynolds number as

$$\lim_{\delta_\tau\to 0}\tilde\epsilon = \tilde\epsilon_0 = k_1\eta\tau^+ \tag{3}$$

for the overlap region Σ.

A solution of the pipe problem consists of integrating Eq. (2) over the inverval $0 < \eta \le 1$ at a fixed δ_τ subject to the boundary condition $u^+ = 0$ at $y^+ = 0$. To carry out such an integration the eddy viscosity as obtained in Eq. (3) for the overlap region must be extended in η to cover the interval $0 \le \eta \le 1$ and in δ_τ to $\delta_\tau > 0$.

2) Core region

The core region is defined as the region between the overlap region and the center line of the pipe. By its nature the overlap region Σ designates the location where viscous effects subside and turbulent effects become predominant. Thus as the overlap region is traversed in the direction of the core, the flowfield becomes more turbulent than viscous in nature and consequently the interaction between turbulent shear stress and viscous shear stress can be expected to decline further, excluding possibly the immediate vicinity of the center line itself. In the limit of small but non zero δ_τ it is therefore assumed that within the core region the turbulent momentum transport coefficient is the same as the turbulent transport coefficient in the overlap region at $\delta_\tau = 0$ viz.,

$$\tilde\epsilon_c = (\tilde\epsilon)_{\delta_\tau=0} = k_1\eta\tau^+ \tag{4}$$

where $\tilde\epsilon_c$ designates $\tilde\epsilon$ in the core region C.

The equation of motion for the core becomes then

$$\tau^+ = (\tilde\epsilon_0 + \delta_\tau)du^+/d\eta \tag{5}$$

which is valid for $0 < \delta_\tau \ll 1$ and $\eta_c < \eta < 1$. Equation (5) states that within the core region the total shear is obtained as a superposition of the turbulent and viscous shear with zero interacting terms. These physical assumptions, once accepted, have the following mathematical implication: Let $\tilde\epsilon(\eta,\delta_\tau)$ have a regular asymptotic expansion in C with respect to δ_τ of the form,

$$\begin{aligned}\tilde\epsilon(\eta,\delta_\tau) &= \tilde\epsilon_0(\eta) + \delta_\epsilon\tilde\epsilon_1(\eta) + \ldots\\ &= \tilde\epsilon_0(\eta) + (\delta_\epsilon/\delta_\tau)\delta_\tau\tilde\epsilon_1(\eta) + \ldots\end{aligned} \tag{6}$$

where δ_ϵ is a small parameter which depends on δ_τ so that $(\delta_\epsilon/\delta_\tau) \to 0$ as $\delta_\tau \to 0$. Substituting Eq. (6) into Eq. (2) yields

$$\tau^+ = [\tilde\epsilon_0(\eta) + (\delta_\epsilon/\delta_\tau)\delta_\tau\tilde\epsilon_1(\eta) + \ldots + \delta_\tau]du^+/d\eta \tag{7}$$

Comparing Eq. (7) and Eq. (5) indicates that the consequence of the assumption made in Eq. (4) is equivalent to assuming $\delta_\epsilon \sim 0(\delta_\tau{}^{1+\alpha})$ where $\alpha > 0$. Carrying the mathematical argument farther, since $\tilde\epsilon$ must depend continuously on δ_τ as η changes from C to Σ one obtains for the overlap region the expansion for nonzero δ_τ in the form,

$$\tilde\epsilon(\eta,\delta_\tau) = \tilde\epsilon_0(\eta) + \delta_\tau{}^{1+\alpha}\tilde\epsilon_1(\eta) + \ldots \tag{8}$$

and consequently the equation of motion

$$\tau^+ = (\tilde{\epsilon}_0 + \delta_\tau)du^+/d\eta + 0(\delta_\tau^{1+\alpha}) \quad (9)$$

in Σ for $0 \leq \delta_\tau \ll 1$.

Integrating Eq. (14) between η and 1 for $\eta > \eta_\Sigma$, with $\tau^+ = 1 - \eta$ yields the velocity defect law in the form,

$$G(\eta,\delta_\tau) = u_{\max}{}^+ - u^+ = \frac{1}{k_1}\frac{1}{\eta_1 - \eta_2} \times \left[\eta_1 \ln\frac{1-\eta_2}{\eta - \eta_2} - \eta_2 \ln\frac{\eta_1 - 1}{\eta_1 - \eta}\right] \quad (10)$$

where $\eta_1 = (\frac{1}{2})[1 + (1 + 4\delta_\tau/k_1)^{1/2}]$ and $\eta_2 = (\frac{1}{2})[1 - (1 + 4\delta_\tau/k_1)^{1/2}]$ involving the single constant k_1. The error in Eq. (10) assuming $\tilde{\epsilon} = \tilde{\epsilon}_0$ throughout the interval of integration is of the order of $\delta_\tau^{1+\alpha}$. In the neighborhood of the center line where the total shear approaches zero the non negative turbulent and viscous shears must independently go to zero. Thus the "non interaction" assumption may very well break down near the center line where turbulent and viscous shear become of comparable order, leading in turn to an error in the value of $u_{\max}{}^+$. This error can be estimated as follows: Let the assumption $\epsilon = \epsilon_0$ in $\eta_\Sigma < \eta \leq 1$ break down at the point η^* where the turbulent and viscous shear become of comparable order viz., $\tilde{\epsilon}_0 du^+/d\eta \sim \delta_\tau du^+/d\eta$. The point η^* is determined then by the solution of the equation

$$\tilde{\epsilon}_0 = k_1\eta\tau^+ = 0(\delta_\tau)$$

which has the two roots $\eta_1{}^* = 1 - 0(\delta_\tau/k_1)$ and $\eta_2{}^* = 0(\delta_\tau/k_1)$. The second root $\eta_2{}^*$ is clearly outside of the core region and need not be considered. Now for the neighborhood of the center line the defect law may always be written as

$$u_{\max}{}^+ - u^+ = -\tfrac{1}{2}(d^2u/d\eta^2)_{\eta=1}(1-\eta)^2 + \ldots$$

where $(du/d\eta)_{\eta=1} = 0$. By Eq. (2) for any eddy viscosity model which has a bounded first derivative at $\eta = 1$, $(d^2u/d\eta^2)_{\eta=1} = -(\tilde{\epsilon}_{\eta=1} + \delta_\tau)^{-1}$ and therefore $|d^2u^+/d\eta_2|_{\eta=1} \leq \delta_\tau^{-1}$ for any $(\epsilon)_{\eta=1} \geq 0$. Thus the maximum error in $u_{\max}{}^+$ brought about by assuming $\tilde{\epsilon} = \tilde{\epsilon}_0$ for $\eta > \eta_1{}^*$ is given by $|u_{\max}{}^+ - u^+(\eta^*)| \leq (\frac{1}{2})\delta_\tau{}^{-1}(1 - \eta_1{}^*)^2 = 0(\delta_\tau)$.

Associating with the location of the overlap region Σ within the flowfield, the neighborhood where the turbulent shear reaches a maximum, one obtains from the condition $(d/d\eta)(\tilde{\epsilon}_0 du^+/d\eta) = 0$ the result,

$$\eta_\Sigma = (\delta_\tau/k_1)^{1/2}[1 - 2(\delta_\tau/k_1)^{1/2}] + 0(\delta_\tau{}^{3/2}) \quad (11)$$

or in the terms of the y^+ coordinate,

$$y_\Sigma{}^+ = (\delta_\tau k_1)^{-1/2}[1 - 2(\delta_\tau/k_1)^{1/2} + 0(\delta_\tau)] \quad (11')$$

The physical significance of Eq. (11) is that when viewed from the core as $\delta_\tau \to 0$ the C region extends to the wall while when viewed from the wall under the same limit the wall region extends to infinity.

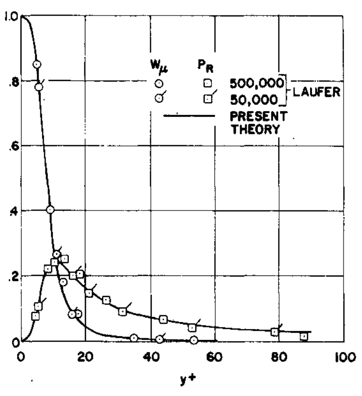

Fig. 1 Turbulent energy production rate and direct viscous dissipation near wall.

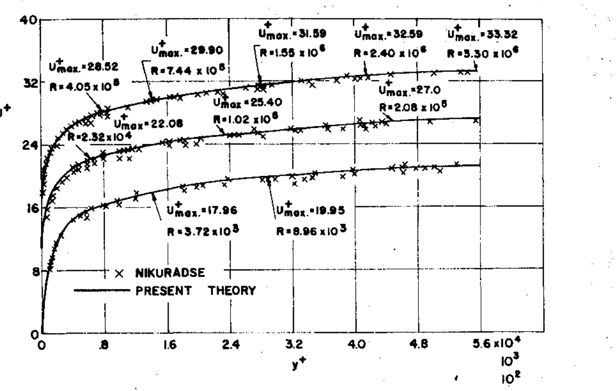

Fig. 2 Mean velocity distributions—Nikuradse.

3) *Wall region*

Following the method used in Ref. (2) the eddy viscosity model for the wall region will be derived here, subject to the asymptotic condition that as the overlap region Σ is approached,

$$\epsilon^+ \to \epsilon_0{}^+ = k_1 y^+\tau^+ \quad (12)$$

For the overlap region, $u^+ = (1/k_1)\ln y^+ + k_2' = (1/k_1)\ln k_2 y^+$ may be written with u^+ as the independent variable as $y^+ = (1/k_2)\exp(k_1u^+)$ and thus ϵ^+ in Σ in terms of u^+ becomes,

$$\epsilon_0{}^+ = k_1(1/k_2)\exp(k_1u^+)\tau^+ \quad (13)$$

The turbulent shear in the immediate vicinity of the wall was shown by Reichardt[3] to vary proportionally to at least the cubic power of the distance from the wall; and since for the same neighborhood the mean velocity u, assuming non zero wall shear, varies linearly with the same dimension it follows from the definition $-\langle\rho u'v'\rangle = \rho\epsilon du/dy$ that

$$\rho\epsilon \sim y^3 \sim u^3 \quad (14)$$

Including a correction function f in Eq. (13), ϵ^+ in W can be written as

$$\epsilon^+ = k_1 S\tau^+ \quad (15)$$

where

$$S = (1/k_2)[\exp(k_1u^+) + f(k_1u^+)] \quad (16)$$

The function S in accordance with Eq. (12) is subject to the asymptotic condition,

$$\lim_{y^+\to y_\Sigma{}^+} S/y^+ \to 1 \quad (17)$$

where $y_\Sigma{}^+$ designates y^+ in Σ; and, in accordance with Eq. (14) it is subject to the boundary conditions implied on S through the relations,

$$\epsilon^+ = d\epsilon^+/du^+ = d^2\epsilon^+/d^2u^+ = 0 \quad (18)$$

The boundary conditions (18) taken in order yield,

$$\begin{aligned} \epsilon^+ &= 0 \quad S(o)\tau^+(o) = 0 \\ (\epsilon^+)' &= 0 \quad S(\tau^+)' + S(o)\tau^+ = 0 \\ (\epsilon^+)'' &= 0 \quad S(\tau^+)'' + 2S'(o)(\tau^+)' + S''(o)\tau^+ = 0 \end{aligned}$$

$$\begin{aligned} S(o) &= 0 \quad f(o) = -1 \\ S'(o) &= 0 \quad f'(o) = -1 \\ S''(o) &= 0 \quad f''(o) = -1 \end{aligned} \quad (19)$$

where primes denote differentiation with respect to k_1u^+.

Taking the highest degree polynomial which satisfied Eq. (19) and at the same time does not introduce new arbitrary constants yields for the function f,

$$f(\xi) = -(1 + \xi + \tfrac{1}{2}\xi^2) \quad (20)$$

and for the function S

$$S(\xi) = k_2^{-1}[\exp(\xi) - (1 + \xi + \tfrac{1}{2}\xi^2)] \qquad (21)$$

where $\xi = k_1 u^+$.

An eddy viscosity model of the form $\epsilon^+ = k_1 S$ for the wall region has been proposed by Spalding.[4] Clearly, Spalding's formula as well as the model proposed by Kleinstein[2] coincide with the present model in the region where $\tau^+ = 1$.

It is apparent that once the asymptotic condition lim $(s/y^+) \to 1$ has been realized in Σ, as y^+ increases this condition continues to hold. An increase in y^+ however implies a farther penetration into the core region, but on the other hand as was shown in the core analysis the model $\epsilon^+ = k_1 y^+ \tau^+$ is valid for $\Sigma + C$. Thus we are led to the conclusion that for sufficiently large Reynolds numbers the eddy viscosity model $\epsilon^+ = k_1 S \tau^+$ holds through the entire pipe. It can be shown (Ref. 5) that "sufficiently large" means $0(\delta_\tau^{1/4} \ln \delta_\tau) \ll 1$.

III. Comparison with Experimental Results

The turbulent pipe flow problem as obtained by the aforementioned analysis is defined by the differential equation

$$du^+/dy^+ = \tau^+/(1 + \epsilon^+) \quad 0 < y^+ \leq \delta_\tau^{-1} \qquad (22)$$

subject to the boundary conditions,

$$u^+ = 0 \quad \text{at} \quad y^+ = 0 \qquad (22a)$$

where $\tau^+ = 1 - y^+\delta_\tau$, $\epsilon^+ = k_1 S \tau^+$, $S = k_2^{-1}[\exp(\xi) - (1 + \xi + \frac{1}{2}\xi^2)]$, $\xi = k_1 u^+$ provided $\lim(S/y^+) \to 1$ as $y^+ \to y_\Sigma^+$.

This theory is complete with the exception of the two arbitary constants k_1 and k_2 which must be determined experimentally. From Laufer's[6] experiment at $R_{max} = (u_{max}D/\nu) = 500,000$, the value of δ_τ as determined from pressure drop and u_{max} measurements was found to be 0.000114. Taking k_1—the von Kármán constant—as 0.4, the value of k_2 is determined in such a way that upon integrating Eq. (22) with $\delta_\tau = 0.000114$ the resulting Reynolds number $R_{max} = (u_{max}D/\nu) = 2u_{max}{}^+\delta_\tau^{-1}$ is equal to 500,000. A calculation with $k_2 = 11.0$ gave $u_{max}{}^+ = 28.69$ and $R_{max} = 5.033 \times 10^5$. With $k_2 = 11.0$ as an accepted value a large number of calculations has been carried out for different values of the parameter δ_τ. In Fig. (1) viscous dissipation $w_\mu = \nu^2 u_\tau^{-4}(du/dy)^2 = (du^+/dy^+)^2$ and turbulent energy production $P_\tau = \epsilon^+ w_\mu$ as computed by the present analysis are compared with Laufer's[6] results at the 50,000 and 500,000 Reynolds numbers. In Fig. (2), the mean velocity distributions for ten different Reynolds numbers are compared with Nikuradse's data as presented in Goldstein.[7] As can be seen from the above figures the agreement with experimental data is good. Additional comparisons for the friction law and turbulent shear distributions could be found in Ref. (5).

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Condensation Augmented Velocity of a Supersonic Stream

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OVER the years, condensation phenomena have been considered as they arose in turbines, in rocket nozzles and in wind tunnels.[1] When condensation occurs, the particles generated as well as the changed conditions of temperature, pressure, etc., result in problems such that, in general, efforts have been made to somehow prevent the condensation. More recently, however, condensation has been suggested as a means of producing particles for colloidal thrustors[2,3] and for EHD propulsion.[4] Presented here is another example of a possible advantage of condensation which has to do with the effect of condensation on velocity in supersonic flow. In previous studies very little attention has been paid to the effects of phase change on the velocity.

When heat is added to a supersonic stream in a constant area duct, the velocity of the stream must decrease.[5] It is simply shown here, however, that if heat is added to a supersonic stream in a diverging section of nozzle, the heat addition can yield an increase in nozzle exit velocity, over that obtained in the same nozzle but without the heat addition. Initially, when the heat is added to the supersonic stream, the velocity decreases. However, as the stream continues to expand, the increase in local temperature which results from the heat addition has a smaller effect on the velocity than the increase in what may be called "effective" stagnation temperature does. The velocity then increases above what it would be without the heat addition. Since nozzle exit velocity is a measure of specific impulse, increased nozzle exit velocity is of particular interest in propulsion schemes.

Adding heat to a supersonic stream without creating large disturbances ordinarily would present some problems. However, transferring the latent heat of vaporization to the stream by condensation can be an effective means of doing just that. One advantage of such heat transfer for a velocity increase is that it can be accomplished without any increase in the temperature which any solid boundary sees, since revaporization occurs when the flow is decelerated near the wall.

When homogeneous condensation occurs in a supersonic nozzle the particles have been shown to be so small[6] that it may be assumed they have the same velocity and temperature as the flowing gas. Then for steady-state one-dimensional flow of an ideal gas with condensation, the energy equation may be written as

$$C_pT + u^2/2 = C_pT_0 + gL \qquad (1)$$

where C_p is the specific heat at constant pressure, (considered constant) T is local temperature, u is local velocity, T_0 is stagnation temperature upstream of the condensation, g is mass fraction of condensed vapor, L is latent heat. gL represents the heat added and $C_pT_0 + gL$ could be regarded as the "effective" stagnation temperature mentioned earlier. The velocity at any station is

$$u = \{2[C_pT_0 + gL - C_pT]\}^{1/2} \qquad (2)$$

For "frozen" flow (without condensation) the velocity would be

$$u_n = \{2[C_pT_0 - C_pT_n]\}^{1/2} \qquad (3)$$

where subscript n refers to the "frozen" flow. It can then be

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